MICRO SERIES

David Tweed

Fundamentals of Second-Order Systems

Part 1: Where It All Began

Part 1 of 3

Understanding second-order systems isn't a mystery, but with so many options and applications, it's hard to get an understanding of the principles. So, Dave kicks off this series by explaining some of the basics.

The second-order system is common in nearly all branches of engineering—car suspensions, robotic servos, audio filters, to name a few—and so, a good understanding of it can be a powerful tool in your arsenal. But, the approach taken in many engineering schools discourages students and leaves them with an understanding that associates black magic with the topic.

In this series, I'll dispel that notion by demonstrating how common second-order systems are and by showing how analogies can be drawn among them. Yes, I'll get into mathematics, even calculus, but I promise that it will be painless. And when you notice how widely applicable the math is, you won't mind at all.

In Part 1, I'm going to talk about the basic second-order system in terms of a physical model (mass-spring-damper) and an electrical model (LCR circuit). You'll learn the equivalencies between these two models and derive the basic mathematics that describe them.

Part 2 will expand the discussion to the use of second-order systems as filters and then move on to cover servomechanisms. You'll learn how negative feedback affects the behavior of the system and extends the mathematical model. Other implementations based on op-amps and DSP (digital signal processing) will be discussed, too.

Part 3 will bring the discussion home with a detailed analysis of some real-world servomechanisms such as a mechanical positioner for a robot and an electronic phase-locked loop. I'll discuss how to predict their performance. How unavoidable nonlinearities affect the behavior of the system and ways to mitigate their effects will be covered.

WEIGHTS, SPRINGS, AND DAMPERS

In general terms, a second-order system is any system that has a measurement that can change. It stores energy related to that change (restoring force), stores energy related to the rate of change (inertia), and dissipates energy and comes to a stop (resistance). Nearly every textbook about this subject starts by discussing the physical system of mass, spring, and damper, so I'll start there.

Figure 1 shows the arrangement; a movable mass is attached to both a spring and dashpot (shock absorber), and the other ends are fixed. The vertical position of the mass is the changeable measurement. The spring provides the restoring force and stores potential energy in proportion to how far the mass is moved from its equilibrium position. The mass stores kinetic energy in proportion to how fast it is moving. The dashpot dissipates energy (turns it into random heat) at a rate proportional to the velocity of the free end. Assume that all of the friction in the system is embodied in the dashpot.

The mass's important parameter is, well, its mass in units of kilograms. (I'll stick to the metric system throughout this series because it re-

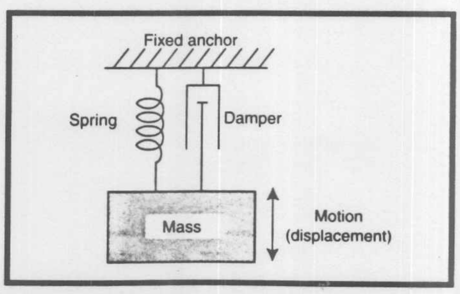

Figure 1—*The basic second-order mechanical system consists of a spring, mass, and damper, or dashpot.*

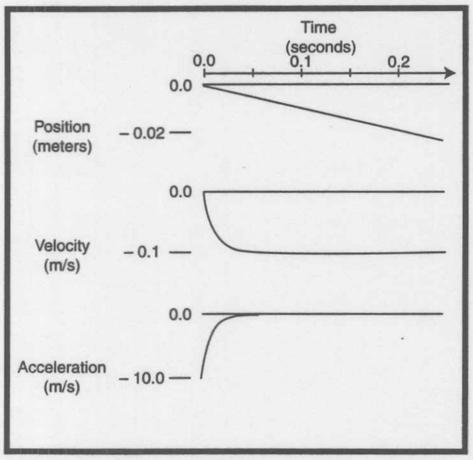

Figure 2—*A system of just a mass and a damper quickly reaches a terminal velocity, where the acceleration has decayed to zero.*

quires fewer conversion constants.) In the earth's gravity field, which has an acceleration of roughly -10 m/s^2, it takes a force of 10 newtons (N) to suspend a mass of 1 kg in midair.

The spring is characterized by its spring constant (K_s), which indicates how much restoring force it develops for a given amount of displacement. The metric units would be newtons per meter. The restoring force is in the opposite direction relative to the displacement; hence, if you move the mass downward, the spring's force is upward, and vice versa.

Assume the spring is –100 N/m. If you hang your 1-kg mass on it, the spring will need to develop a 10-N upward force to counteract the –10-N (downward) force of gravity. As a result, it will stretch by:

$$\text{displacement} = \frac{\text{force}}{K_S} = \frac{10\text{ N}}{-100\text{ N/m}} = -0.1\text{ m} = -10\text{ cm}$$

The dashpot also develops a negative force, but this force is proportional to the velocity of the free end rather than its position. The dashpot is characterized by the constant K_d in newtons per meter per second. The definition of "work" is a force operating over a distance. It has the same units as energy, which are joules (J), and 1 J is defined as a force of 1 N operating over a distance of 1 meter, or 1 newton-meter.

The work associated with moving the free end of the dashpot against its force is turned into randomized heat and is essentially lost to the system. This is in contrast to the spring, which stores that work as potential energy and returns it to the system when the movement reverses direction. Let's play with different values of K_d and determine what effect this has on the response of the system.

Imagine the dashpot filled with thick, syrupy oil. It requires a great force to move it even slowly; it has a high K_d, say, 10 N/m/s. If you hang the 1-kg mass from the dashpot, it quickly reaches a terminal velocity of –0.1 m/s. The work, or energy, needed to move the dashpot comes from the fact that the mass is descending in earth's gravitational field. Terminal velocity occurs when the two forces balance; the 10 N of gravitational force on the mass is balanced by 10 N of upward force from the dashpot when the velocity is –0.1 m/s.

A SIMPLE SIMULATION

Let's put together a mathematical simulation. I use MathCad because it makes it easy to see the math and graphics at the same time, but you can use a spreadsheet (see "Simulating in a Spreadsheet" sidebar) or your favorite programming language.

The general technique is simple: set up three variables that represent position, velocity, and acceleration, set their initial values, and write simple equations that update the three values after a short increment of time. After a few hundred (or a few thousand) timesteps, you can graph the evolution of the system by graphing the three values. In MathCad, use a vector to hold the three values so that you write a single equation to do the updates for each timestep.

Start by entering the system parameters (gravitational acceleration, mass (M), and damper parameter, respectively):

$$G = -10 \times \frac{m}{s^2}$$

$$M = 1 \times kg$$

$$K_d = -10 \times \frac{N}{\left(\frac{m}{s}\right)}$$

Also, set your timestep and number of steps and set up an index variable. The following equations are timestep, number of steps, and index variable, respectively:

$$dT = 0.001 \times s$$
$$N = 200$$
$$I = 1 \ldots n$$

The next step is to enter the initial conditions (position, velocity, acceleration, respectively):

$$p_0 = 0.1 \times m$$

$$v_0 = 0 \times \frac{m}{s}$$

$$A_0 = G$$

Next, enter the expressions that control updating. Position is easy, it gets incremented by the previous velocity value multiplied by the timestep. Similarly, velocity gets incremented by the previous acceleration value multiplied by the timestep. Acceleration is trickier, it's the result of the net sum of the forces (in this case, the constant acceleration of gravity and the damper force that is proportional to the velocity) divided by the mass. Use vector notation to ensure that all three values get updated at each timestep before moving on to the next, putting the position on top, velocity in the middle, and acceleration (a) on the bottom:

$$\begin{bmatrix} p_i \\ v_i \\ a_i \end{bmatrix} = \begin{bmatrix} p_{i-1} + dT \times v_{i-1} \\ v_{i-1} + dT \times a_{i-1} \\ G + \dfrac{Kd \times v_{i-1}}{M} \end{bmatrix}$$

Figure 2 shows the results, with the mass quickly reaching a constant terminal velocity when the force of the dashpot balances that of gravity. You can also think of this in terms of energy flow; terminal velocity occurs when the work being put into the

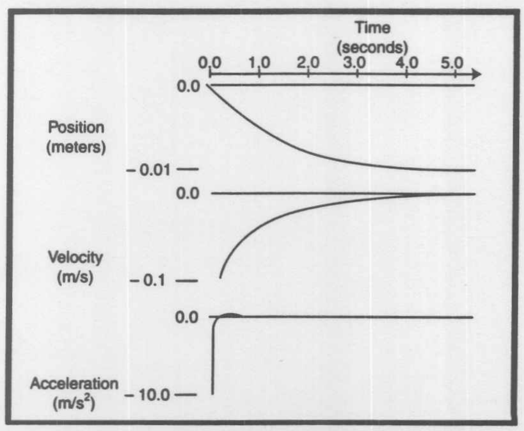

Figure 3—*An overdamped system responds to disturbances with a slowly decaying exponential movement.*

Physical system		Electrical system	
Item	**Units**	**Item**	**Units**
Position of mass	Meters	Capacitor charge	Coulombs
Velocity of mass	Meters/second	Current	Amperes (coulombs/second)
Physical force	Newtons	Electromotive force	Volts
Physical work	Newton-meters	Electrical work	Volt-coulombs
Mass	Kilograms	Coil	Henries
Spring	Newtons/meter	Capacitor	1/farads (volts/coulomb)
Dashpot	Newtons/meter/second	Resistor	Ohms (volts/coulomb/second)

Table 1—*There is a direct analogy for each parameter of the physical system in the electrical system.*

system by the gravity field equals the energy being dissipated by the dashpot as heat. Remember this point, because I'll come back to it later.

Now, let's go back to the system that includes the spring. It has an equilibrium point defined by where the weight of the mass is balanced by the upward pull of the stretched spring. If you lift up the mass from this point and then release it, it will gradually return to that equilibrium point. The acceleration of the mass is no longer constant, because as it moves down, the spring exerts an increasing upward force that partially cancels the force caused by gravity.

The other upward force on the mass comes from the resistance of the dashpot. The work being done on the mass by the spring (and gravity) must equal the work being done on the dashpot by the mass and getting dissipated. But, the dashpot's force is proportional to the velocity of the mass and the spring's force is proportional to the position of the mass. The net force on the mass is proportional to its acceleration.

So, the position, velocity (time derivative of the position), and acceleration (time derivative of the velocity) of the mass decay to zero in proportion to each other. One curve whose derivative varies in proportion to itself is the exponential (the general shape of the path followed by the mass).

To see the effect of adding the spring, add its parameter at the top of the simulation you did previously and modify the updated equation to calculate the spring parameter:

$$K_s = -100 \times \frac{N}{m}$$

$$\begin{bmatrix} p_i \\ v_i \\ a_i \end{bmatrix} = \begin{bmatrix} p_{i-1} + dT \times v_{i-1} \\ v_{i-1} + dT \times a_{i-1} \\ G + \dfrac{K_d \times v_i - 1 + K_s \times p_{i-1}}{M} \end{bmatrix}$$

You'll need more steps in order to see everything else, so change N to 5000 (see Figure 3).

RESONANCE

Now imagine a dashpot that's filled with only air with K_d of 2. If you change the parameter in the simulation and rerun it, you get the behavior

shown in Figure 4. Because the dashpot provides so little force, the mass has quite a bit of momentum built up when it reaches the equilibrium position, so it keeps going until the increasing force from the spring becomes enough to stop it.

However, now it's too low, so it starts back in the other direction. This process keeps repeating at a regular rate, called the natural, or resonant, frequency of the system. A system configured this way is said to be underdamped. The oscillations do die out eventually, because the dashpot still is dissipating energy from the system.

The resonant frequency (F_0) is primarily determined by the mass and spring (although, as I'll explain later, the dashpot does affect it). A higher K_s will raise the frequency and a larger mass will lower it. Ignoring K_d for the moment, the resonant frequency is:

$$F_0 = \frac{1}{2\pi}\sqrt{\left(\frac{|K_S|}{M}\right)}$$

With the values just mentioned, this equals 1.592 Hz. In fact, you can see about eight cycles in the 5-s elapsed time of the simulation.

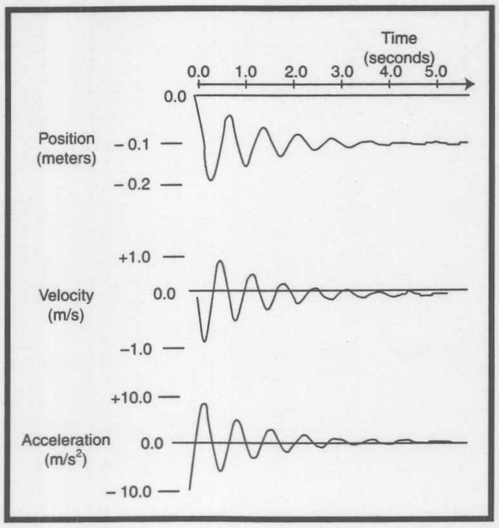

Figure 4—*An underdamped system oscillates in response to disturbances at a particular resonant frequency. This oscillation dies out exponentially.*

You can imagine a case somewhere between these two extremes of K_d. At some point, as the resistance of the dashpot is gradually decreasing, the system's response is just about to become periodic. In other words, there's a definite boundary between the overdamped and underdamped regimes. When the system is in this configuration, it is said to be critically damped. I'll leave this subject for now, but I'll return to it later.

COILS, CAPACITORS, AND RESISTORS

Now let's take a look at the canonical electrical second-order system, which consists of a coil, capacitor, and resistor (see Figure 5). The coil is equivalent to the mass, the capacitor is equivalent to the spring, and the resistor corresponds to the dashpot. In fact, for every parameter or quantity in the physical system, there's a corresponding one in the electrical system (see Table 1).

Start by defining the electrical equivalent of position of the mass to be charge (on the capacitor), which is measured in coulombs. This means that the electrical equivalent of velocity, or meters per second, is coulombs per second, or amperes (A). This seems natural enough.

Next, define the electrical equivalent of mechanical force, or newtons, to be electromotive force, or volts. Using both of these definitions, the mechanical concept of work = force × distance leads to the electrical concept of work = volts × coulombs. Following this to its natural conclusion, power is defined as work per unit of time. If you write power = volts × coulombs/seconds and note that cou-

Simulating in a Spreadsheet

Even if you don't have a mathematical modeling software package like MathCad, you can do this type of simulation with a spreadsheet program. I'm going to briefly outline the general technique, which should work with nearly any spreadsheet program.

The idea is to dedicate one column of the spreadsheet to each variable in the simulation. In the top row, label a column for each variable. Put the initial value of each variable in the next row, below its label. It's a good idea to have a column that represents time, with an initial value of zero. The other columns will represent acceleration, velocity, position, and so on.

After you have written the mathematical expressions that update the variables, enter them as formulae into the third row of the spreadsheet and use the values of the system variables from the row above. The update formula for time is simply the previous value plus a constant that represents the time interval for each step of the simulation.

Here's the trick: using whatever mechanism your spreadsheet uses, select the cells in the third row and replicate them downward, using as many rows as the number of timesteps you want to simulate. Make sure that the cell references for system variables remain relative, always using values from the row immediately above.

Now you should see a huge field of numbers that show the evolution of the system over time. If your spreadsheet supports the generation of graphs (nearly all of them do), you can graph the columns representing the system variables against the column representing time, and you should get graphs similar to the ones that MathCad generates.

If you want to change the initial conditions, modify the values in the second row of the spreadsheet. If you want to change the update formulas, do that in the third row, but then you'll need to replicate the cells below the one(s) you change again.

You can avoid this second replication if you know how to generate absolute as well as relative cell references in your formulae. Instead of putting constants into the formulae, replace them with absolute references to cells near the top of the spreadsheet. Then if you want to change the parameters of the simulation, you can just change the contents of those cells and the rest of the spreadsheet will update itself.

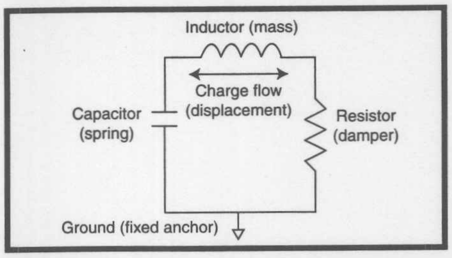

Figure 5—*The basic second-order electrical system consists of a capacitor, coil, and resistor.*

lombs/seconds = amperes, you get power = volts × amperes, which we're all familiar with.

The remaining equivalencies among the mechanical components and their electrical counterparts may look strange at first, but let's work through the details.

The coil, whose inductance is measured in henries (H), is equivalent to the mass, which is measured in kilograms. Using the relationship F = M × a, kilograms can be thought of as F/a, or newtons per meter per second squared. By definition, a coil is a device that develops a voltage across itself in proportion to the rate at which the current is changing, which means that henries can be thought of as volts per ampere per second, or volts per coulomb per second squared. You end up with the corresponding fundamental units you started with in corresponding positions in the two expressions.

The electrical capacitor is equivalent to the mechanical spring. You measured K_s in units of newtons per meter. In a capacitor, Q (charge) = C (capacitance) × V (voltage), so C = Q/V. In order to establish the equivalence between K_s and the capacitor, you need to use 1/C, or units of 1/farads. In effect, a stiff spring (large K_s) corresponds to a tiny capacitor (low capacitance).

Finally, the electrical resistor corresponds to the mechanical dashpot. Using the relationship V = I × R, ohms can be thought of as V/I, or volts per coulomb per second. Again, the units match up in the expressions for the two systems.

Everything I said about the mechanical system also applies to the electrical system when you substitute the correct quantities and units. The first simulation you did corresponds to a circuit with just a coil and resistor. The force of gravity is equivalent to putting a battery in series, therefore, you can see how the terminal velocity of the mechanical system corresponds to the VDC current that flows in the R–L circuit.

For the second simulation, you put the capacitor back in the circuit. It now limits the flow of charge in the same way that the spring limits movement. And in the third simulation, lowering K_d is equivalent to lowering R, and then the electrical circuit rings. When you change the units in the formula for the resonant frequency, it becomes:

$$F_0 = \frac{1}{2\pi}\sqrt{\left(\frac{1}{L \times C}\right)} = \frac{1}{2\pi \times \sqrt{L \times C}}$$

SHAKE, RATTLE, AND ROLL

The next question is, what happens if you apply a continuous stimulus to either the physical system or electrical system, instead of just nudging it and watching what happens? For a number of reasons, the sinewave is the simplest form of continuous motion, so let's study that. Fourier states that you can make any other periodic motion by adding sinewaves, which makes it a good place to start.

Start with the physical system. You can apply a stimulus by making the fixed anchor movable and then either applying a force to it or moving it a specific amount. In either case, moving the anchor requires that an entity outside of the system do work—an energy input to the system. The only energy output from the system still occurs via the damper (or resistor). Therefore, it seems obvious that the system response will move toward a condition in which the dissipation of the damper equals the input energy.

Clearly, both the spring and dashpot will attempt to transfer any motion of the anchor to the mass, whose inertia will resist it, causing the spring and dashpot to change length. If you do this slowly, the spring and dashpot don't change length by much; instead, the entire system, including the mass, simply moves up and down in sync with the anchor.

On the other hand, if you do it quickly, the inertia of the mass holds it nearly still and all of the motion is

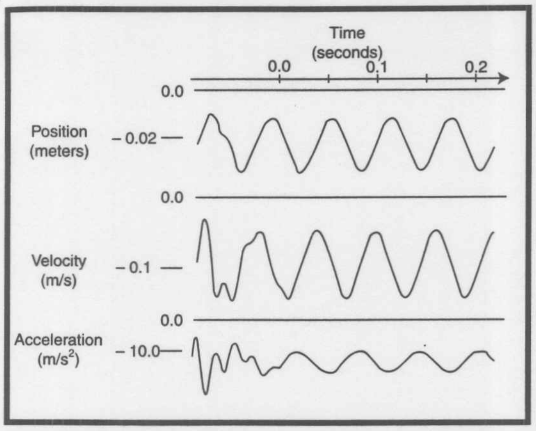

Figure 6—*The system response under continuous stimulation shows both a transient part and steady-state part.*

taken by the spring and dashpot. Between these two extremes is where more interesting things happen.

Applying a force to the anchor is equivalent to a series voltage source. Actually moving the anchor is equivalent to using a current source in parallel with the inductor. The analogy is that if the mass (coil) is large, it moves (carries current) hardly at all, while all of the motion (current) is taken by the spring (capacitor) and damper (resistor).

If the frequency of the voltage or current source is low, the voltage (and charge) on the capacitor will track that of the source closely. But if the frequency is high, it will change little.

In an overdamped system with a large K_d, most of the force applied to the anchor will be coupled with the mass via the damper rather than the spring. The electrical analogy is that the high R, rather than the impedance of the capacitance, limits the movement of charge through the inductor and the rest of the circuit.

In the underdamped system, different things happen depending on whether the applied frequency is less than, equal to, or greater than the system's resonant frequency. Let's update the simulation to incorporate this external force.

An external force applied to the anchor point gets added to the other forces acting on the mass. Set the initial position equal to the equilibrium position to simplify the interpretation of the results (see Figure 6).

The first three equations are frequency, anchor force, and initial position, respectively:

$$F = 0.5 \times Hz$$

$$\mathrm{Force}(t) = \sin(2 \times \pi \times F \times t)N$$

$$p_0 = -\frac{G \times M}{K_s}$$

$$\begin{bmatrix} p_i \\ v_i \\ a_i \end{bmatrix} = \begin{bmatrix} p_{i-1} + dT \times v_{i-1} \\ v_{i-1} + dT \times a_{i-1} \\ G + \dfrac{K_d \times v_{i-1} + K_s \times p_{i-1} + \mathrm{Force}(dT \times (i-1))}{M} \end{bmatrix}$$

You can see irregularities in the graphs near the beginning, before things settle down to a regular pattern. Looking at the acceleration graph, you can see that the transient part has a frequency component that is similar to the natural frequency of the system calculated earlier and an shape that resembles the graphs in Figure 4. This resonance is excited by the fact that the simulation applies the acceleration suddenly at time zero (a phenomenon known as jerk). After this transient response dies, the only frequency component present is the 0.5 Hz of the stimulation.

UNTIL NEXT TIME

You've observed the analysis of both the transient and steady state responses of some simple second-order systems. It's been an interesting mental exercise, but I'm sure you're wondering how all of this can be applied in the real world. Well, don't worry, that's what I'll get into in Part 2.

Dave Tweed is an independent consultant. He has been developing hardware and real-time software for microprocessors for many years. His system design experience includes computer design from supercomputers to workstations, digital telecommunications systems, and the application of embedded microcomputers and DSPs. You may reach him at dtweed@acm.org.

SOFTWARE

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MICRO SERIES

David Tweed

Fundamentals of Second-Order Systems

Part 2: The Tools of the Trade

Part 2 of 3

If you dig equations then you'll enjoy this article. If understanding how mathematical tools affect second-order systems is a source of much headscratching for you, dig in and enjoy the insight Dave offers here.

Welcome to Part 2. I'm presenting some of the common theoretical background and mathematical tools for many of the dynamic systems found in all branches of engineering. These include filters of various types and servomechanisms such as phase-locked loops and robotic positioners.

Part 1 covered the basic setup of both mechanical and electrical second-order systems. The former consists of a mass, spring, and damper, and the latter consists of a coil, capacitor, and resistor. Every quantity in each system has an analogue in the other, and the same math is used to describe both systems. A simple numerical simulation was also set up.

Now it's time to take a deeper look at the underlying mathematics and derive more powerful ways of describing such systems. We'll take a look at how these systems can be used as filters, and then what happens when negative feedback is added to the system, as in servomechanisms.

I'm not going to spend a lot of time talking about the Mathcad simulations, but there are files that you can download and play with for most of the figures in this installment.

MATHEMATICAL MODEL

The mathematical formulae you built so far are just stepwise numerical simulations. But, what about a closed-form, continuous solution? It turns out not to be too difficult, and involves only a little calculus. Bear with me and we'll get through it quickly.

Start by listing the constraints of the components and constraints arising from how they're connected. The coil develops a voltage in proportion to its inductance and the time derivative of the current through it:

$$V_{coil} = L\frac{dI}{dt}$$

The capacitor carries a current that is proportional to its capacitance and the time derivative of the voltage across it:

$$I = C\frac{dV_{cap}}{dt}$$

Or equivalently, the capacitor voltage is the integral of the current through it divided by the capacitance:

$$V_{cap} = \frac{1}{C}\int I \times dt$$

The resistor develops a voltage in proportion to its resistance and the current through it:

$$V_{res} = R \times I$$

The current, *I*, is the same everywhere in the circuit.

The voltages must sum to zero around the circuit:

$$V_{cap} + V_{res} + V_{coil} = 0$$

Or, after making these substitutions:

$$\frac{1}{C}\int I \times dt + R \times I + L\frac{dI}{dt} = 0$$

This final equation is the basic second-order problem. If both sides of it are differentiated once with respect to time, you get a linear, second-order homogeneous, differential equation:

$$\frac{1}{C} I + R\frac{dI}{dt} + L\frac{d^2I}{dt^2} = 0$$

Solving such equations is difficult. However, note that here you need a function that, when added to its first and second derivatives, sums to zero.

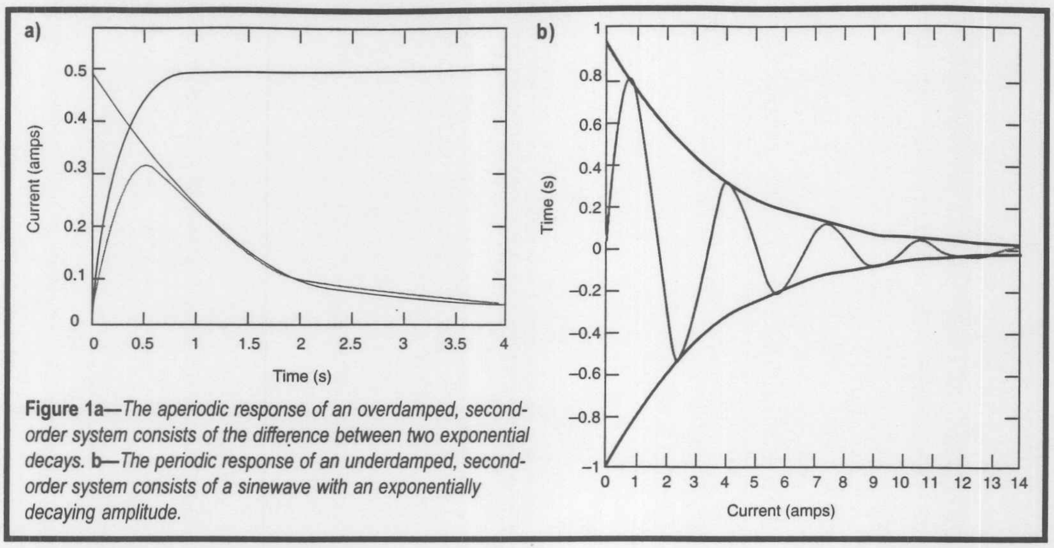


Figure 1a—*The aperiodic response of an overdamped, second-order system consists of the difference between two exponential decays.* **b**—*The periodic response of an underdamped, second-order system consists of a sinewave with an exponentially decaying amplitude.*

The solution probably will be an exponential function. Let's try:

$$i(t) = Ae^{st}$$

The first and second derivatives are:

$$\frac{di(t)}{dt} = Ase^{st}$$

$$\frac{d^2i(t)}{dt^2} = As^2e^{st}$$

Making the substitutions, the differential equation becomes:

$$\frac{1}{C}Ae^{st} + RAse^{st} + LAs^2e^{st} = 0$$

$$Ae^{st}\left(\frac{1}{C} + Rs + Ls^2\right) = 0$$

Solving this requires that the third factor in parentheses be set to zero. Because this is a quadratic equation, there are two solutions:

$$s = \frac{-R - \sqrt{\frac{R^2 - 4L}{C}}}{2L}$$

$$s = -\frac{R}{2L} + \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}$$

By defining a couple of new constants, you can write a simpler form:

$$\alpha = \frac{R}{2L} \text{ and } \omega_0 = \frac{1}{\sqrt{LC}}$$

$$s = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

α is the damping coefficient, and ω_0 is the resonant frequency. In general, ω is used to denote a radian frequency, which is related to a frequency f in hertz by a factor of 2π. I'm going to stick with radian frequency to keep the formulae simple; if you want to convert to hertz, just use the relation:

$$f = \frac{\omega}{2\pi}$$

You'll also find it useful to define the damping ratio:

$$\frac{\alpha}{\omega_0} = \zeta$$

Look at the argument to the square root: If this quantity is negative, then the square root is an imaginary number and the solution s is going to be a conjugate pair of complex numbers. In real terms, this means that the decay is periodic. If it's exactly zero, then the circuit is said to be critically damped. And if the quantity is positive, there will be two real values of s in the solution and the decay will be aperiodic. These three cases correspond to the quantity ζ being less than, exactly, and greater than 1, respectively.

It can be useful to solve this expression for how R relates to L and C (see initial quadratic solution):

- if $R^2 = 4L/C$, the circuit is critically damped
- if $R^2 \geq 4L/C$, the decay is aperiodic
- if $R^2 < 4L/C$, the decay is periodic

The full form of the solution can be written out like this:

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

$$i(t) = A_1e^{s_1t} + A_2e^{s_2t}$$

A_1 and A_2 are picked to satisfy whatever initial conditions are set up for the circuit. If s_1 and s_2 are two different real values, the overall function is the sum of two exponential equations with two different time constants.

In general, you'll find that A_1 and A_2 have equal values but opposite signs, so you end up with a function that is zero at t = 0 and at t = infinity (see Figure 1a).

If s_1 and s_2 are a complex conjugate pair, you can define a different frequency, the natural frequency:

$$\omega_N = \sqrt{\omega_0^2 - \alpha^2}$$

And, this allows you to write the solution as:

$$i(t) = A_1e^{(-\alpha + j\omega_N)t} + A_2e^{(-\alpha - j\omega_N)t}$$

Euler's Identity says:

$$E^{jKt} = \cos Kt + j \sin Kt$$

Remembering that $A_1 = -A_2$, you find that the cosine terms cancel and you're left with an expression that describes a decaying sinewave:

$$i(t) = 2A_1e^{-\alpha t}j \sin(\omega Nt)$$

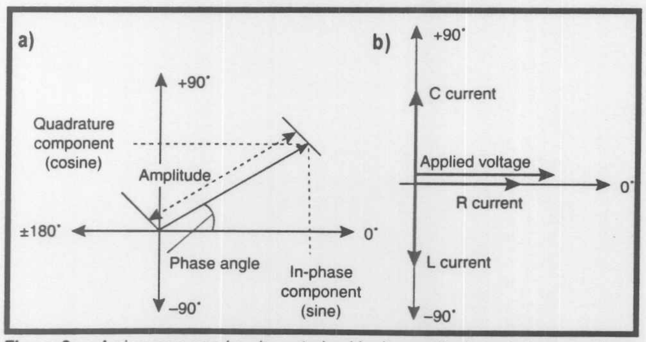


Figure 2a—*A sinewave can be characterized by its amplitude and phase angle, or equivalently, by the magnitudes of its in-phase (sine) and quadrature (cosine) components. This vector can represent any sinewave quantity at a particular frequency.* **b**—*Current is in phase with the applied voltage in a resistor, but it lags by 90° in a coil and leads by 90° in a capacitor.*

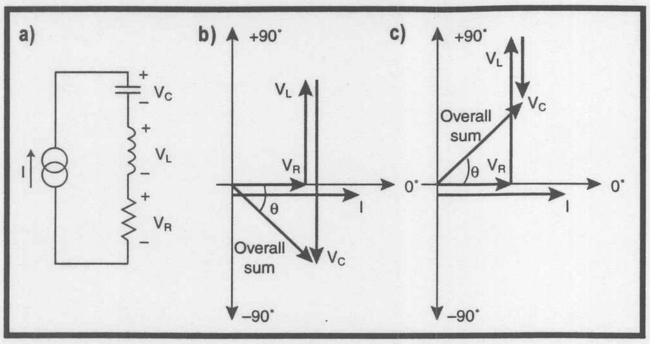


Figure 3a—*Here's the RLC circuit with a current source added in series. To find the total voltage, the component voltages must be added vectorially by placing the individual vectors end-to-end.* **b**—*At frequencies below resonance, the capacitive component is larger than the inductive component, resulting in an overall negative phase shift θ.* **c**—*At frequencies above resonance, the overall phase shift θ is positive.*

Figure 1b shows an example of this type of response.

The Mathcad sheet for Figure 1 lets you plug in values for R, L, and C to see the effect on α, ω_0, ω_N, and ζ.

DRIVEN RESPONSE

This information provides insight into the basic mechanisms of second-order systems, but you're probably more interested in the response of the system to periodic driving forces. You did a simulation last month, now it's time to derive the math that describes this aspect of the system.

This requires a different set of math tools from those used so far. Here, phasors and complex impedances rule the day, but first let me get a few basic concepts out of the way.

Remember that the fast Fourier transform shows that any periodic forcing function can be expressed as a sum of sines and cosines, or equivalently, as a sum of sines whose relative phase can be varied. Because you're dealing with strictly linear circuits here, you can use the principle of superposition to analyze the circuit response one sinewave component at a time. And you can derive the overall response by summing the components at the end. Also, because of the linearity, the forced response must be a sinewave at the same frequency, but with a different amplitude and/or phase.

Together these facts mean that if you understand the response of a circuit or system to sinewaves of various frequencies, you always can get the response to arbitrary periodic forcing functions. Any sinewave is characterized by just two parameters, its amplitude and initial phase. Together, these parameters can be thought of as a vector in phase space, also known as a phasor (see Figure 2a).

Amplitude can be in units of any physical quantity. This can let you show the relationship between voltage and current for your components (see Figure 2b). In the resistor, the current always is in phase with the applied voltage; or, if you apply a current, the voltage is always in phase with that. The relationship between voltage and frequency is independent of the frequency.

In the coil, the current always lags the voltage by 90°, or equivalently, the voltage leads the current by 90°.

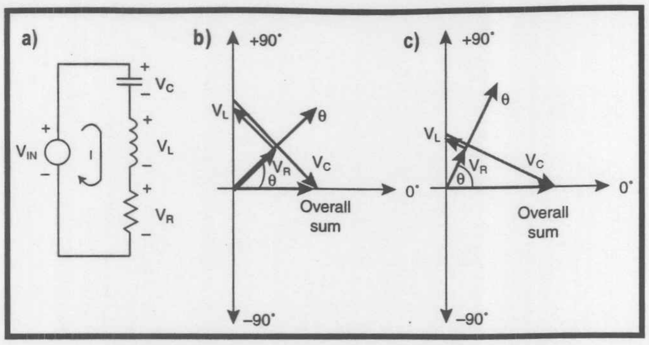


Figure 4a—*If the current source is replaced by a voltage source, the component relationships remain the same, but the diagrams must be rotated to put the overall voltage on the 0° axis.* **b**—*Below resonance, the current has a positive phase shift θ relative to the applied voltage.* **c**—*At a lower frequency, the current is smaller and the phase shift θ is greater.*

Look at Figure 3a. Because the current is the same everywhere in a series circuit, each of the components has a voltage across it determined by the effect of the current on that component. The voltage across the current source is the sum of the three component voltages. Figure 3b shows how they add up. Because the voltages are at different phase angles, use vector addition to add them. The frequency of the current source is less than the resonant frequency of the circuit. The capacitor's voltage is greater than the coil's voltage, so the overall sum has a phase angle that's negative with respect to the current source (which is arbitrarily set at 0° for simplicity).

As the frequency is decreased further, the C component continues to grow while the L component shrinks, causing the total vector to lengthen and the phase angle to increase. In the limit, the vector is infinitely long and parallel to the –90° axis. This corresponds to a DC current source charging the capacitor indefinitely.

Figure 3c shows the case where the source frequency is greater than the resonant frequency. Now, the coil's

However, their magnitudes are related by a function that depends on frequency. If the voltage is held constant, then the current decreases with increasing frequency, or if the current is held constant, the voltage increases with increasing frequency. Because impedance is defined to be the ratio of voltage to current, the last statement is the same as saying the impedance of the coil increases with frequency.

Everything is reversed for the capacitor. Thus, the voltage always lags the current by 90°, or equivalently, the current leads the voltage by 90°. Again though, their magnitudes are related by a function that depends on frequency. However, now, if the voltage is held constant, the current escalates with increasing frequency, or if the current is held constant, the voltage (and impedance) decreases with increasing frequency.

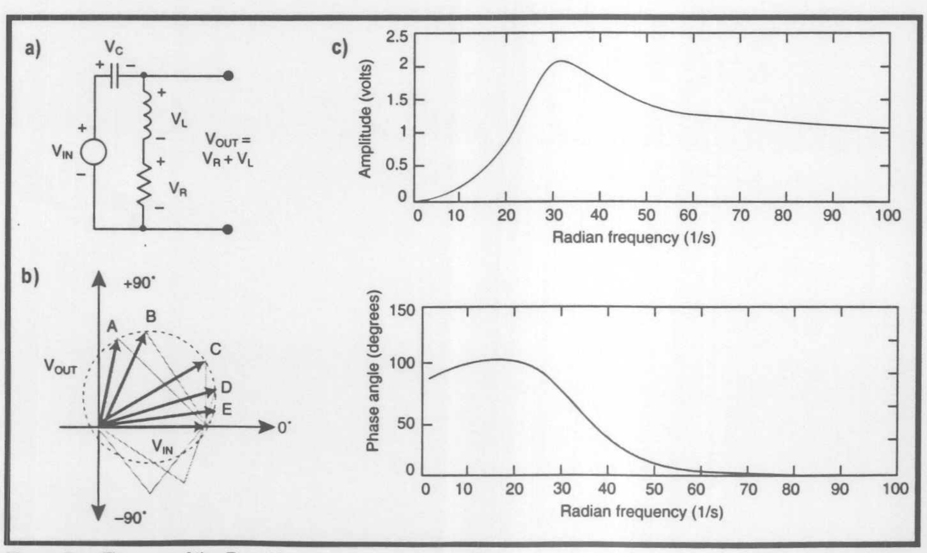
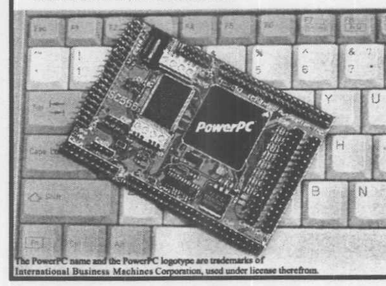
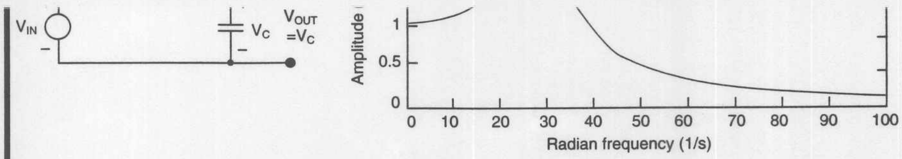


Figure 5a—*The sum of the R and L voltage components produces a high-pass filter function.* **b**—*The output voltage and phase angle are given by the vector showing the partial sum of the R and L components.* **c**—*When the magnitude and angle of the R + L vector is plotted against frequency, you see the high-pass function.*





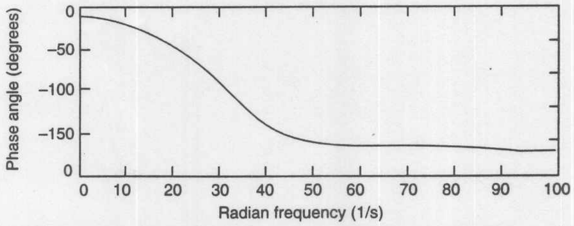


Figure 6a—*Swapping the R and L with the C produces a low-pass filter function.* **b**—*The magnitude of the C vector is the complement of the high-pass response, and the phase angles have the opposite values.*

voltage component is greater than the capacitor's component (i.e., the coil's impedance is greater than the capacitor's, because the current is the same in both) and the phase angle of the sum is positive. If the frequency continues to increase, both the phase angle and amplitude increase, as well.

As you might have guessed, the resonant case is the frequency at which the voltage (and impedance) of the coil and capacitor cancel each other, leaving only the resistive component in the sum. At this point, the total voltage is at a minimum and the phase angle is 0°, which means that the voltage is in phase with the current.

What happens if a voltage source is substituted for the current source (see Figure 4a)? Well, you have the same sum of voltages as before, but now the total voltage is constrained to match that of the voltage source and the current phase angle and magnitude are free to vary.

Figure 4b shows what happens when Figure 3b is rotated so that the sum voltage lies along the 0° axis, scaled so that the length of this vector corresponds to the applied voltage. The current is still the same everywhere in the circuit, and because it is in phase with and proportional to the voltage across the resistor, just look at the resistive component of the voltage diagram to see what it is.

The Mathcad sheet for Figures 3 and 4 lets you to plug in various values for R, L, C, and ω to see the effect on the various component vectors.

SECOND-ORDER FILTERS

Second-order systems can be used as filters of various types. As you saw, the series RLC circuit has a notch in its voltage response when driven with a current source and a peak in its current response when driven with a voltage source. This gives basic band-stop and band-pass capabilities, but what about high-pass and low-pass functions? Start by redrawing the circuit of Figure 4a (see Figure 5a), and take a look at the sum of the resistive and inductive components of the response. This is simply a matter of drawing an additional vector in the phase sketch, as shown in Figure 5b, which shows the R + L component for several different frequencies.

The curved, dotted line shows the locus of points reached by this vector for a wide range of frequencies. Note that ω_0 corresponds to point C on this curve, but this isn't where the V_{OUT} vector is the longest, which actually occurs at point D. Point D corresponds to the ω_N calculated earlier.

Figure 5c shows the length (magnitude) and direction (phase angle) of the R + L vector plotted as a function of frequency. And sure enough, it looks like a high-pass filter.

Go back to the physical mass-spring-damper system, where the spring corresponds to the capacitor, the mass to the coil, and the damper to the resistor. Recall that current in the electrical system corresponds to the motion of the mass (relative to the anchor point) in the physical sys-

tem and that voltage corresponds to physical force. The AC voltage source in the electrical system is equivalent to a cyclical mechanical force applied to the anchor point of the mechanical system (which is free to move).

If the cyclical frequency is high, the mass tends to sit still because of its inertia. The output voltage of the electrical system is equivalent to the total mechanical force across the damper and the net force on the mass. The damper resists high-speed motion, so it couples most of the input force directly to the mass, and the anchor point moves very little. This means that there is insignificant force across the spring, which requires a strong physical motion of one end or the other to develop significant force. Any motion of the mass and anchor point will be in phase with each other and with the applied force.

If the cyclical frequency is low, the mass moves in response to the forces on it, and because the damper provides little resistance to this low-velocity motion, most of this force is coupled by the spring. The anchor point moves as well, and the magnitude of the motion of the mass relative to the anchor point (corresponding to current) is small. The damper plus mass force (the analog of the electrical output voltage) is negligible.

The low-pass function is constructed by realizing that whatever input voltage doesn't appear across the R-L combination must be across the C. So, if the R-L combination is swapped with the C (see Figure 6a), you get the inverse response shown in Figure 6b. You can do the same sort of vector analysis for the high-pass filter, but then the output vector is the same as the C component alone.

UNTIL NEXT TIME

Now you've seen some of the math that describes the transient behavior and steady state of second-order systems. If you'd like to learn more, read the book listed in the Resources. It covers the topic with more detail and rigor, but in a very readable way.

Next month, I'll talk about active second-order systems and then move onto the topic of servomechanisms.

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W. Hayt, Jr., and J. Kemmerly, *Engineering Circuit Analysis (McGraw Hill Series in Electrical and Computer Engineering)*, McGraw-Hill Higher Education, Burr Ridge, IL, 1978.